An exact relation between number of black box computations required to solve an oracle problem quantumly and quantum retrocausality

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Abstract

We investigate the reason for the quantum computational speedup – quantum algorithms requiring fewer computation steps than their classical counterparts. We extend the representation of the quantum algorithm to the process of setting the problem – of choosing the function computed by the black box. The initial measurement selects a setting at random, Bob (the problem setter) unitarily changes it into the desired one. This representation is to Bob and any external observer, it cannot be to Alice (the problem solver). It would tell her the function computed by the black box, which to her should be hidden inside it. We resort to the observer dependent quantum states of relational quantum mechanics. To Alice, the projection of the quantum state due to the initial measurement is retarded at the end of her problem solving action. To her, the algorithm input state remains one of complete ignorance of the setting. By black box computations, she unitarily sends it into the output state that, for each possible setting, encodes the corresponding solution, acquired by the final measurement. We show that there can be a quantum feedback from final to initial measurement. We can ascribe to the final measurement the selection of any part – say the $\mathcal{R}$-th part – of the random outcome of the initial measurement. This projects the input state to Alice on a state of lower entropy where she knows a corresponding part of the problem setting. The quantum algorithm turns out to be a sum over classical histories in each of which Alice, knowing in advance one of the $\mathcal{R}$-th parts of the setting, performs the black box computations still required to identify the solution. Given an oracle problem and a value of $\mathcal{R}$, this retrocausality model provides the number of black box computations required to solve it quantumly. Conversely, given a known quantum algorithm, it yields the value of $\mathcal{R}$ that explains its speed up. In the major quantum algorithms, $\mathcal{R}$ is $\frac{1}{2}$ or slightly above it. $\mathcal{R} = \frac{1}{2}$ always yields the number of black box computations required by an existing quantum algorithm and the order of magnitude of the number required by optimal one.

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1 Foreword

Consider the following problem. Bob, the problem setter, chooses one of the four functions $f_{\mathbf{b}}(\mathbf{a})$ whose tables are given in array (1):

$\mathbf{a}$	$f_{00}(\mathbf{a})$	$f_{01}(\mathbf{a})$	$f_{10}(\mathbf{a})$	$f_{11}(\mathbf{a})$
0	0	0	1	1
1	0	1	0	1

(1)

Then he gives Alice, the problem solver, a black box (*oracle*) that, given a value of the argument $\mathbf{a}$ in the input, produces the value of $f_{\mathbf{b}}(\mathbf{a})$ in the output. Alice does not know which of the four functions is the one computed by the black box. She is to determine whether the function is constant or balanced (ie with the same number of zeros and ones) by performing function evaluations (*oracle queries*). Classically, Alice must perform two function evaluations, quantumly just one. We are speaking of the seminal quantum algorithm that yields a *quantum computational speedup*, devised by Deutsch [1] in 1985.

Although there is a significant body of literature on the relationship between speedup and other quantum features, such as *quantum entanglement* and *discord* (see Section 3), no fundamental physical explanation nor unified mathematical mechanism is known for the speedup.

As the present issue of Quanta is dedicated to Feynman and our subject is quantum computation, we should like to remember Feynman's pioneering contribution to the development of this new branch of science.

We do this by recalling the seminal works that gave rise to the discipline of quantum computation. In 1969, Finkelstein [2] noted that computation should be possible in the quantum framework and introduced the notion of quantum unit of information, namely quantum bit or *qubit*. In 1982 Feynman [3] pointed out the essential difference between quantum and classical computation, showing that the simulation of a quantum process on a classical computer has to involve in general an amount of time×physical resources exponentially higher than that involved in the quantum process itself. This was the origin of the notion of quantum computational efficiency, now usually called speedup. Always in [3], Feynman introduced the universal quantum simulator, a lattice of spin systems with freely specifiable nearest neighbor interactions that can be considered the first theoretical model of a quantum computer. The development of the notion of reversible classical computation was parallel. Bennett [4] showed in 1982 that classical computation can be ideally reversible in the limit of zero speed. His work was in the wake of the 1961 *Landauer's principle* [5] that quantifies the generation of heat necessarily consequent to the erasure of information. Still in 1982, Fredkin and Toffoli [6] developed the first theoretical model of logically reversible classical computation, in fact based on the well known Fredkin and Toffoli gates. Independently of Deutsch, Feynman produced in 1985 the quantum version of this algorithmic form of reversible computation, published in the following year [7]. The seminal idea of it was already present in [3], in fact with reference to Fredkin&Toffoli 1982 work. We already cited

the 1985 Deutsch's paper with the first example of a quantum algorithm that requires fewer function evaluations than classically possible.

With this, the full fledged notion of quantum computation was born thanks to the insights of very few individuals. As it might happen with revolutionary science, the scientific community at large has been initially slow in expressing an interest for the new discipline. We had the fortune of contributing, with Mario Rasetti, to the organization of the first international meetings on quantum communication and computation held in Turin in the years 1992÷1998. They were the Elsag Bailey (an Italian ICT company)-ISI (Institute for Scientific Interchange) workshops. We believe that those annual workshops have been instrumental to propagating the new discipline of quantum information throughout the scientific community. All the fathers of the discipline, with the sad exception of Feynman, the theoretical and experimental physicists and computer scientists responsible for the major developments of those years attended the workshops in question. The group pictures of reference [8] show the explosion of interest for the new science in the years 1993 through 1997.

2 Introduction

The usual physical representation of quantum algorithms is limited to the process of solving the problem. We extend it to the process of setting the problem, namely of choosing the function $f_{\mathbf{b}}(\mathbf{a})$ out of the set of functions. This amounts to choosing the function suffix $\mathbf{b}$, which we call *the problem setting*, out of the set of the possible problem settings $\sigma_B \equiv \{00, 01, 10, 11\}$ – we use Deutsch algorithm as an example.

For reasons that will soon become clear, we assume that the initial state of the quantum register B that contains the setting is a mixture of all the possible settings. Its density operator is thus $\rho_B = \frac{1}{4}\left(|00\rangle\langle 00|_B + |01\rangle\langle 01|_B + |10\rangle\langle 10|_B + |11\rangle\langle 11|_B\right)$ – a few mathematical representations make things more clear.

At time t_0, Bob measures the content of register B, obtaining a setting at random, say $\mathbf{b} = 10$. The state of register B is consequently projected on $|10\rangle\langle 10|_B$. Assume that Bob wants $\mathbf{b} = 01$. He unitarily transforms this state into $|01\rangle\langle 01|_B$, at time t_1.

Register A, meant to contain the argument of the function to be computed by the black box and eventually the solution of the problem, at time t_1 is in any sharp state, say $|0\rangle\langle 0|_A$. The input state of the quantum algorithm at time t_1 is thus $|01\rangle\langle 01|_B \otimes |0\rangle\langle 0|_A$.

Alice, with one function evaluation preceded and followed by suitable transformations, unitarily transforms it into the output state $|01\rangle\langle 01|_B \otimes |1\rangle\langle 1|_A$, at time t_2. The solution of the problem, 1 when the function is balanced as in the present case and 0 when it is constant, is in register A. Alice acquires the solution by measuring the content of A.

We note that this extended representation immediately calls for another extension, this time concerning the actors (observers) on the stage. We have to resort to the relational quantum mechanics of Rovelli [9], where quantum states

are observer dependent. A quantum state can be sharp to an observer and a quantum superposition, or a mixture, to another one. The present representation is with respect to Bob, the problem setter, and any other observer who does not act on the problem solving process. It cannot be with respect to Alice, the problem solver. The input state of the quantum algorithm $|01\rangle\langle 01|_B \otimes |0\rangle\langle 0|_A$ would of course tell us that the content of register B is 01, namely that the function chosen by Bob is $f_{01}(\mathbf{a})$. Since Alice is an observer, we assume that the state in question would tell her the same.

Throughout this work, we take for granted the legitimacy of the assumption that Alice (or Bob, or the external observer), although an abstract entity, knows what we would know in her (his) place. We assume that this is a legitimate way to take the sizes to a quantum process that necessarily involves the notion of observer.

In the present case, Alice would know that the function is balanced without performing any function evaluation. Of course the suffix of the function should be hidden to Alice – to her it is inside the black box.

We physically represent this concealment by retarding the projection of the quantum state induced by the initial Bob's measurement at the end of the unitary part of Alice's problem solving action. As well known, these projections can be retarded or advanced at will along a unitary evolution that respectively follows or precedes the measurement.

The input state of the quantum algorithm to Alice, immediately after the preparation of the desired problem setting, remains thus $\frac{1}{4}(|00\rangle\langle 00|_B + |01\rangle\langle 01|_B + ...) \otimes |0\rangle\langle 0|_A$. In fact the maximally mixed state of register B remains unaltered under any unitary transformation applying to it. The two bit entropy of this state represents Alice's complete ignorance of Bob's choice.

The output state to Alice is $\frac{1}{4}(|00\rangle\langle 00|_B \otimes |0\rangle\langle 0|_A + |01\rangle\langle 01|_B \otimes |1\rangle\langle 1|_A + ...)$, namely still a mixture of all the possible problem settings, each multiplied by the corresponding solution. Thus, also the solution, considered in itself, is completely undetermined. Alice's final measurement projects this state on the solution corresponding to the problem setting chosen by Bob, namely on $\frac{1}{2}(|01\rangle\langle 01|_B + |10\rangle\langle 10|_B) \otimes |1\rangle\langle 1|_A$, with probability one. In fact the solution is unpredictable to Alice but is already 1 to any other observer.

Alice's final measurement also triggers the retarded projection induced by the initial Bob's measurement, which cannot go past the unitary part of Alice's action. This further projects the above state on $|01\rangle\langle 01|_B \otimes |1\rangle\langle 1|_A$, which tells Alice both the problem setting and the solution. The two projections commute and should be considered simultaneous.

We note that either the projection of the quantum state induced by the initial Bob's measurement or that induced by the final Alice's measurement zeroes the entropy of the solution, depending on which one is performed first. This work is an exploration of the assumption that this zeroing shares in a complementary and non-redundant way between initial and final measurement.

We assume that the complete measurements behave in a contextual way – each would be sensitive to the other. We assume that they reduce (in all the possible ways in quantum superposition as we will see) to partial measurements

such that, together, select whatever has been selected by the complete measurements and, each by itself, reduce the entropy of the solution in a complementary and non-redundant way. For Occam razor, we should exclude any redundancy. This implies that the information provided by either partial measurement is not provided by the other.

To reconstruct the selections performed by the complete measurements, we should propagate forward in time, along the time-forward unitary transformation, the projection of the quantum state due to the partial Bob's measurement, until it selects part of the outcome of Alice's measurement. Similarly, we should propagate backward in time, along the inverse of the time-forward unitary transformation, the projection due to Alice's partial measurement, until it selects part of the random outcome of Bob's measurement.

We will see that everything boils down to ascribing to the final Alice's measurement the selection of part of the random outcome of the initial Bob's measurement, say the $\mathcal{R}$-th part of the information that specifies it.

This *quantum feedback* leaves the input state of the quantum algorithm to Bob and any external observer unaltered. It projects that to Alice on a state of lower entropy where she knows the $\mathcal{R}$-th part of the information that specifies the problem setting, before performing any function evaluation. Alice uses this knowledge to solve the problem with fewer function evaluations. In other words, sharing the selection of the final measurement outcome between initial and final measurement explains the speedup.

There are many ways of taking the part of a whole. The quantum algorithm turns out to be a sum over classical histories in each of which Alice knows in advance one of the possible $\mathcal{R}$-th parts of the problem setting and performs the function evaluations still necessary to find the solution of the problem.

We can speak of Alice's *advanced knowledge* of part of the problem setting because, in the absence of quantum retrocausality, Alice would know the setting only with the final measurement. Moreover this knowledge comes to her from her final reading of the solution.

Given an oracle problem and a value of $\mathcal{R}$, the present retrocausal explanation of the speedup yields a number of function evaluations required to solve the problem quantumly. Conversely, given a known quantum algorithm, it yields the value of $\mathcal{R}$ that explains its speedup.

We have compared this model with the major quantum algorithms discovered so far. In all the quantum algorithms that solve the problem with a single function evaluation, as that of Deutsch, we have $\mathcal{R} = \frac{1}{2}$. This also applies to Grover quantum search algorithm for database size 4, Deutsch & Jozsa algorithm, and the algorithms of Simon and the Abelian hidden subgroup. The latter algorithm [10] has unified about ten historical algorithms, among which the famous Shor's factorization algorithm. In Grover algorithm, when database size goes past 4, first $\mathcal{R}$ goes slightly above $\frac{1}{2}$ then it goes back to $\frac{1}{2}$ for database size tending to infinity.

In the corresponding sample of problems, $\mathcal{R} = \frac{1}{2}$ always corresponds to an existing quantum algorithm and yields the order of magnitude of the number of function evaluations required to solve the problem in an optimal quantum way.

If this held in general, we would have a very powerful tool, the way of assessing the order of magnitude of the number of function evaluations (oracle queries) required to solve a generic oracle problem in an optimal quantum way.

3 Positioning the work

The present work is the further development of the approach [11 ÷ 13]. We have further clarified the retrocausal explanation of the speedup and developed a procedure for computing the number of function evaluations required to solve a generic oracle problem with quantum retrocausality $\mathcal{R} = \frac{1}{2}$.

The present explanation of the speedup is in line with the tenet of time-symmetric quantum mechanics of Aharonov et al. [14, 15, 16, 17], which states that the complete description of the quantum process between initial and final measurement requires knowledge not only of the outcome of the initial measurement, also of that of the final one. This naturally implies that the latter outcome has back in time implications on the upstream process. As a matter of fact, the form of quantum retrocausality utilized in the present work has been inspired by the work of Dolev and Elitzur [17] on the non-sequential behavior of the wave function highlighted by partial measurement.

The work has points of contact with works of Morikoshi. In [18], this author highlights the problem-solution symmetry of Grover's and the phase estimation algorithms and notes it may be relevant for the explanation of the speedup. In [19], he shows that Grover algorithm violates a temporal Bell inequality. There should be a connection between this violation and the form of quantum retrocausality we are dealing with.

Besides [11 ÷ 13], we are not aware of literature relating the speedup to quantum retrocausality. There are of course other approaches to the problem of unifying the explanation of the speedup. Reference [20] (2001) shows that the presence of multipartite entanglement with number of parties increasing unbounded with problem size is necessary for achieving exponential speedup in pure state quantum computing. It also conjectures that there could be exponential speedup in the absence of entanglement in mixed state quantum computing.

The notion of quantum discord was introduced independently in [21] and [22]. Discord is a measure of non-classical correlations between two subsystems of a quantum system that are not necessarily entangled – it coincides with entanglement in pure state quantum computing. It could be of high practical interest, since it shows the possibility of achieving a speedup in mixed state quantum computing – the realistic form of computation in the presence of noise. Reference [23] shows that, contrary to the topical thought at the time, quantum states can be too entangled to be useful for the purpose of computation.

At present, no single reason behind the speedup was found from the standpoint of entanglement and discord. The speedup appears to always depend on the exact nature of the problem while the reason for it varies from problem to problem [22]. The relation between number of oracle queries and quantum retocausality highlighted in the present work appears to hold exactly for any

oracle problem.

We should also cite tree size complexity [24] and contextually based [25] arguments. In the former, a measure of the complexity of the multiqubit state is shown to be related to the speedup of a variety of quantum algorithms. The latter addresses the relation between speedup and the contextual character of quantum mechanics. It identifies a form of fault tolerant quantum computation (by *magic states*) specially resilient to noise. Also the present retrocausal explanation of the speedup could be considered a contextually based argument. The reduction of the initial and final measurements of a quantum process to partial non-redundant measurements is of course contextual in character.

4 The seminal Deutsch algorithm

Let us review the usual representation of Deutsch algorithm, limited to the process of solving the problem. We need two quantum registers: A, of basis vectors $|0\rangle_A$ and $|1\rangle_A$, and V, of basis vectors $|0\rangle_V$ and $|1\rangle_V$. We use ket vectors instead of density operators as in the original Deutsch algorithm.

Bob chooses one of the four functions in array (1), say $f_{01}(\mathbf{a})$, and gives Alice the black box that computes it. Alice knows array (1) but does not know which is the function chosen by Bob. She is to find whether it is constant or balanced through function evaluations. She prepares register A with the value of $\mathbf{a}$ for which she wants to perform function evaluation. The black box computes the value of $f_{01}(\mathbf{a})$ and adds it module two to the former content of register V. Being logically reversible, module two addition can be implemented unitarily. In the introduction we omitted register V because transformations are unitary also without it, but they are more difficult to explain.

For reasons that will soon become clear, the input state of the quantum algorithm is:

$$|\psi\rangle = \frac{1}{\sqrt{2}} |0\rangle_A (|0\rangle_V - |1\rangle_V).$$

Alice applies to register A the Hadamard transform H_A, which transforms $|0\rangle_A$ into $\frac{1}{\sqrt{2}}(|0\rangle_A + |1\rangle_A)$ and $|1\rangle_A$ into $\frac{1}{\sqrt{2}}(|0\rangle_A - |1\rangle_A)$, producing the state:

$$H_A |\psi\rangle = \frac{1}{2} (|0\rangle_A + |1\rangle_A)(|0\rangle_V - |1\rangle_V), \tag{2}$$

then asks the black box to compute the value of the function. Let U_f be the corresponding unitary transformation (defined in the Hilbert space of all registers). We have:

$$U_f H_A |\psi\rangle = \frac{1}{2} (|0\rangle_A - |1\rangle_A)(|0\rangle_V - |1\rangle_V).$$

Function evaluation is performed in quantum parallelism for each term of the input state superposition. It leaves the term $|0\rangle_A (|0\rangle_V - |1\rangle_V)$, appearing in the input state (2), unaltered. In fact, here the argument of the function, the content of register A, is 0. The computation of $f_{01}(0)$ yields 0 that module two added to the former content of register V leaves everything unaltered. Function

evaluation instead changes the term $|1\rangle_A (|0\rangle_V - |1\rangle_V)$ into $|1\rangle_A (|1\rangle_V - |0\rangle_V) = -|1\rangle_A (|0\rangle_V - |1\rangle_V)$. In fact now we have to module two add $f_{01}(1) = 1$ and this changes $|0\rangle_V$ into $|1\rangle_V$ and $|1\rangle_V$ into $|0\rangle_V$.

Then Alice applies a second time the Hadamard transform to register A, obtaining the output state:

$$H_A U_f H_A |\psi\rangle = \frac{1}{\sqrt{2}} |1\rangle_A (|0\rangle_V - |1\rangle_V). \tag{3}$$

Eventually she measures the *content* of register A, namely the observable $\hat{A}$ of eigenstates $|0\rangle_A$ and $|1\rangle_A$ and eigenvalues respectively 0 and 1. She reads the eigenvalue 1, which tells her that the function is balanced (the final content of register A is 0 when the function is constant and 1 when it is balanced).

Thus the problem of checking whether the function given by Bob is constant or balanced is always solved with just one function evaluation quantumly, against two classically.

The mathematics of this speedup, namely that of the quantum algorithm, is obvious in the sense that we have it under the eyes. However, the mathematics of different quantum algorithms are different from one another as there is no known universal scheme. The *mechanism* of the speedups, provided there is one, is not known.

4.1 Time-symmetric and relativized representations

To start with, we extend the representation of Deutsch algorithm to the process of choosing the black box. To this end, we should add an imaginary quantum register B of basis vectors $|00\rangle_B$, $|01\rangle_B$, $|10\rangle_B$, and $|11\rangle_B$. This register contains the problem setting, namely the suffix $\mathbf{b}$ of the function chosen by Bob. The previous black box, which computed $f_{\mathbf{b}}(\mathbf{a})$ for a well determined value of $\mathbf{b}$ and any value of $\mathbf{a}$, is replaced by a universal one that computes $f_{\mathbf{b}}(\mathbf{a})$ for any values of $\mathbf{b}$ and $\mathbf{a}$. Register A and V have the same role as before.

For reasons that will soon become clear, we assume that register B is initially in the maximally mixed state:

$$\rho_B = \frac{1}{4} (|00\rangle\langle 00|_B + |01\rangle\langle 01|_B + |10\rangle\langle 10|_B + |11\rangle\langle 11|_B).$$

As we will need a detailed representation of quantum states and operators, for reasons of encumbrance we represent all states as ket vectors, not matrices. To this end, we move to the random phase representation [26] of the maximally mixed state of register B:

$$|\psi\rangle_B = \frac{1}{2} \left(e^{i\varphi_0} |00\rangle_B + e^{i\varphi_1} |01\rangle_B + e^{i\varphi_2} |10\rangle_B + e^{i\varphi_3} |11\rangle_B \right), \tag{4}$$

where the φ_i are independent random phases with uniform distribution in $[0, 2\pi]$. We will be dealing with a trivial application of the random phase representation: we can always think that the quantum state evolves as a pure state with the

φ_i fixed phases. Only when we have to compute its von Neumann entropy, we should remember that the φ_i are random variables. The von Neumann entropy of state (4), as that of ρ_B, is two bit.

By the way, ρ_B is the average over all φ_i of the product of the ket by the bra: $\rho_B = \langle |\psi\rangle_B \langle\psi|_B \rangle_{\forall\varphi_i}$; reading state (4) is also simple: it is a mixture of pure states with the phases φ_0, φ_1, φ_2, φ_3 all different, in fact a dephased quantum superposition.

The overall initial state of the three registers, at time t_0, is thus:

$$|\psi\rangle = \frac{1}{2\sqrt{2}} \left(\mathrm{e}^{i\varphi_0} |00\rangle_B + \mathrm{e}^{i\varphi_1} |01\rangle_B + \mathrm{e}^{i\varphi_2} |10\rangle_B + \mathrm{e}^{i\varphi_3} |11\rangle_B \right) |0\rangle_A \left(|0\rangle_V - |1\rangle_V \right). \tag{5}$$

In order to prepare register B in the desired problem setting, at time t_0 Bob measures its content, namely the observable $\hat{B}$ of eigenstates the basis vectors $|00\rangle_B, |01\rangle_B, ...$ and eigenvalues respectively $00, 01, ...$ Note that $\hat{B}$ commutes with $\hat{A}$. The measurement outcome is completely random. Say it comes out the eigenvalue $\mathbf{b} = 10$. The state immediately after measurement is:

$$P_B |\psi\rangle = \frac{1}{\sqrt{2}} |10\rangle_B |00\rangle_A \left(|0\rangle_V - |1\rangle_V \right), \tag{6}$$

where P_B is the projection of the quantum state induced by Bob's measurement. Then Bob applies to register B a unitary transformation U_B that changes the random measurement outcome into the desired problem setting, say $\mathbf{b} = 01$. At time t_1 we will have:

$$U_B P_B |\psi\rangle = \frac{1}{\sqrt{2}} |01\rangle_B |0\rangle_A \left(|0\rangle_V - |1\rangle_V \right). \tag{7}$$

State (7) is the input state of the quantum algorithm in the representation extended to the process of setting the problem. There are of course many U_B that change $|10\rangle_B$ into $|01\rangle_B$. For simplicity of exposition, we choose the one that bit by bit changes zeros into ones and ones into zeros:

$$U_B \equiv |11\rangle \langle 00|_B + |10\rangle \langle 01|_B + |01\rangle \langle 10|_B + |00\rangle \langle 11|_B .$$

The output state of the extended representation of the quantum algorithm is:

$$H_A U_f H_A U_B P_B |\psi\rangle = \frac{1}{\sqrt{2}} |01\rangle_B |1\rangle_A \left(|0\rangle_V - |1\rangle_V \right). \tag{8}$$

Of course, input and output states are the same as in the usual representation of the quantum algorithm up to the presence of the ket $|01\rangle_B$.

We note that this extension immediately calls for another one, this time concerning the actors (observers) on the stage. We have to resort to the relational quantum mechanics of Rovelli [10], where quantum states are observer dependent. State (7) is with respect to Bob, the problem setter, and any other observer who does not act on the problem solving process. It cannot be with

respect to Alice, the problem solver. The sharp state $|01\rangle_B$ would tell her, before she starts her search for the solution, that the function chosen by Bob is $f_{01}(\mathbf{a})$. She would know that it is balanced without performing any function evaluation. Of course the suffix of the function should be hidden to Alice – to her it is inside the black box.

To physically represent this fact, it suffices to retard the projection P_B until the end of the unitary part of Alice's action, at time t_2.

To her, the state of register B in the input state of the quantum algorithm is still maximally mixed. In fact U_B leaves state (5) unaltered up to an irrelevant permutation of the independent random phases. Thus, disregarding the permutation, state (5) is the input state to Alice.

We started with register B in a maximally mixed state to represent the fact that, to Alice, the problem setting is physically hidden.

Summing up, states (5) through (8) are the representation of the quantum algorithm with respect to Bob. In the representation with respect to Alice, the input state, which coincides with the initial state, is:

$$U_B|\psi\rangle = |\psi\rangle = \frac{1}{2\sqrt{2}}\left(e^{i\varphi_0}|00\rangle_B + e^{i\varphi_1}|01\rangle_B + e^{i\varphi_2}|10\rangle_B + e^{i\varphi_3}|11\rangle_B\right)|0\rangle_A(|0\rangle_V - |1\rangle_V). \tag{9}$$

The two bit entropy of the state of register B represents Alice's complete ignorance of the problem setting. The output state is:

$$\begin{aligned} H_A U_f H_A U_B|\psi\rangle &= \frac{1}{2\sqrt{2}}\left[\left(e^{i\varphi_0}|00\rangle_B + e^{i\varphi_3}|11\rangle_B\right)|0\rangle_A + \left(e^{i\varphi_1}|01\rangle_B - e^{i\varphi_2}|10\rangle_B\right)|1\rangle_A\right] \\ &\quad (|0\rangle_V - |1\rangle_V), \end{aligned} \tag{10}$$

We can see that, for each possible problem setting (value of $\mathbf{b}$ contained in register B), Alice has built the corresponding solution of the problem $s(\mathbf{b})$ in register A.

Eventually, at time t_2, she acquires the solution by reading the content of register A, namely by measuring $\hat{A}$. We should keep in mind that the output state (10) is with respect to Alice. The same state with respect to Bob and any other observer is $\frac{1}{\sqrt{2}}|01\rangle_B|1\rangle_A(|0\rangle_V - |1\rangle_V)$. The measurement outcome is unpredictable to Alice, it is already 1 to any other observer. Thus Alice's measurement must select the eigenvalue 1 with probability one, projecting state (10) on

$$P_A H_A U_f H_A U_B|\psi\rangle = \frac{1}{\sqrt{2}}\left(e^{i\varphi_1}|01\rangle_B - e^{i\varphi_2}|10\rangle_B\right)|1\rangle_A(|0\rangle_V - |1\rangle_V), \tag{11}$$

where P_A is the projection induced by the final Alice's measurement. State (11) is further projected on:

$$\frac{1}{\sqrt{2}}|01\rangle_B|1\rangle_A(|0\rangle_V - |1\rangle_V) \tag{12}$$

by the retarded projection induced by the initial Bob's measurement. We note that inverting the order of the two projections leaves the end result unaltered. As a matter of fact, since the projection due to Bob's measurement cannot be retarded beyond the unitary part of Alice's action, we should see the two projections as simultaneous. In this way Alice, by measuring $\hat{A}$, also acquires the content of register B. In fact state (12), with register B in the sharp state $|01\rangle_B$, tells Alice that the problem setting chosen by Bob is $\mathbf{b} = 01$. Note that this state is common to both representations, to Alice and to Bob.

In view of what will follow, we note that Alice's measurement of $\hat{A}$ in the output state relativized to her is equivalent to the measurement of $\hat{B}$. In fact either measurement projects state (10) on (12), where the sharp states of registers B and A tell Alice both the setting and the solution of the problem.

4.2 Quantum feedback

We consider the random phase representation of the reduced density operator of register A in the output state (10):

$$|\psi\rangle_A = \frac{1}{\sqrt{2}} \left(\mathrm{e}^{i\Phi_0} |0\rangle_A + \mathrm{e}^{i\Phi_1} |1\rangle_A \right), \tag{13}$$

where Φ_0 and Φ_1 are independent random phases with uniform distribution in $[0, 2\pi]$. The usual representation is $\rho_A = \frac{1}{2} (|0\rangle_A \langle 0|_A + |1\rangle_A \langle 1|_A)$.

$\mathcal{E}_A$, the entropy of $|\psi\rangle_A$, is 1 bit. The zeroing of $\mathcal{E}_A$ can be due to either the projection of the quantum state associated with the measurement of $\hat{B}$ in the initial state (5), retarded at the end of the unitary part of Alice's action, or that associated with the measurement of $\hat{A}$ in the output state (10) (we have seen that the two projections should be considered simultaneous). The present work is an exploration of the assumption that the zeroing of $\mathcal{E}_A$ shares between the two measurements.

To this end, we assume that the two complete measurements reduce to partial measurements such that: **1** together, they select whatever was selected by the complete measurements and **2** each performed alone, contribute in a complementary and non-redundant way to the zeroing of $\mathcal{E}_A$. By this we mean that no information provided by either partial measurement is provided by the other.

We call **1** and **2** *Occam conditions*. They can be seen as an application of Occam razor. In Newton's formulation, it states: *We are to admit no more causes of natural things than such that are both true and sufficient to explain their appearances*. [27]. Here, the razor should exclude any redundancy between initial and final measurement.

The assumption that the two partial measurements contribute equally to the zeroing of $\mathcal{E}_A$, namely that $R = \frac{1}{2}$, explains the speedup of the present quantum algorithm.

We should reduce the initial Bob's measurement and the final Alice's measurement to two partial measurements submitted to **1** and **2** and the condition of equally contributing to the zeroing of $\mathcal{E}_A$.

We have seen that the measurement of $\hat{A}$ in the relativized output state (10) is equivalent to that of $\hat{B}$. Thus we can move to the problem of reducing two measurements of $\hat{B}$, one performed by Bob in the initial state (5) and the other by Alice in the output state (10), to two partial measurements, say of B_i and $\hat{B}_j$, satisfying the above said conditions. In the most general terms, B_i and $\hat{B}_j$ are Boolean functions of $\hat{B}$, such as: $\hat{B}_0$, the content of the left cell of register B, $\hat{B}_1$, the content of the right cell, XOR $\left(\hat{B}_0, \hat{B}_1\right)$, the exclusive or between the two former contents, etc.

We provide an example of reduction of the complete measurements to such partial measurements. We keep the assumption that the initial measurement of $\hat{B}$ randomly selects the eigenvalue $\mathbf{b} = 10$ and that Bob, by U_B, changes it into $\mathbf{b} = 01$. Let $\mathbf{b} \equiv b_0 b_1$; we assume that the eigenvalue $b_0 = 1$ is selected at time t_0 by the measurement of $\hat{B}_0$ in the initial state and that the eigenvalue $b_1 = 1$ is selected at time t_2 by the measurement of $\hat{B}_1$ in the output state.

To reconstruct the selections performed by the complete measurements, we should propagate forward in time, by $H_A U_f H_A U_B$, the projection induced by the former measurement and backward in time, by its inverse, the projection induced by the latter measurement. The two propagations can be performed in any order, the reconstruction is the same.

Let us perform the backward propagation first. The measurement of B_1 in the output state (10), which assumedly selects $b_1 = 1$, projects this state on:

$$|\chi\rangle = \frac{1}{2}\left(\mathrm{e}^{i\varphi_1}|01\rangle_B|1\rangle_A + \mathrm{e}^{i\varphi_3}|11\rangle_B|0\rangle_A\right)\left(|0\rangle_V - |1\rangle_V\right). \tag{14}$$

We advance at time t_0 the two ends of this projection. The result is the projection of the initial state (5) on:

$$U_B^\dagger H_A^\dagger U_f^\dagger H_A^\dagger |\chi\rangle = \frac{1}{2}\left(\mathrm{e}^{i\varphi_3}|00\rangle_B + \mathrm{e}^{i\varphi_1}|10\rangle_B\right)|0\rangle_A\left(|0\rangle_V - |1\rangle_V\right). \tag{15}$$

The permutation of the independent random phases is irrelevant. At this point the measurement of $\hat{B}_0$ in state (15), which assumedly selects $b_0 = 1$, projects it on:

$$|\xi\rangle = \frac{1}{\sqrt{2}}|10\rangle_B|0\rangle_A\left(|0\rangle_V - 1_V\right). \tag{16}$$

Of course state (16), under $H_A U_f H_A U_B$, evolves into state (12), namely $\frac{1}{\sqrt{2}}|01\rangle_B|1\rangle_A\left(|0\rangle_V - |1\rangle_V\right)$, the final state common to both representations (to Bob and to Alice). We have reconstructed the selections performed by the complete measurements. Furthermore, the reduction of $\mathcal{E}_A$ induced by either partial measurement, performed alone, is half bit and no information acquired by either partial measurement is acquired by the other. Conditions **1** and **2** are satisfied.

One can see that, eventually, everything boils down to ascribing the selection of one of the two bits (the right one in present assumptions) of the random outcome of the initial measurement to the final measurement. We are not sending a

message backward in time. Each of the bits that specify the outcome of the initial measurement is independently and randomly selected. We are just ascribing half of these random selections to the final rather than the initial measurement.

We note that sharing between Bob's and Alice's measurements the zeroing of $\mathcal{E}_A$ does not affect Bob's freedom of choosing the function computed by the black box. We should keep in mind that the probability that Alice's measurement of $\hat{B}$ in state (10) selects $\mathbf{b} = 01$, or that the measurement of $\hat{B}_1$ selects $b_1 = 1$ (the right digit of 01), is one. This means that the measurement of $\hat{B}_1$ just reads the right digit of the problem setting $\mathbf{b} = 01$ freely chosen (determined) by Bob, without possibly altering it, or affecting Bob's freedom of choosing it. This goes along with the fact that the backward propagation of the projection due to the measurement of $\hat{B}_1$ in the output state does not determine any part of Bob's choice, but the right digit of the random outcome of Bob's measurement $\mathbf{b} = 10$, which is before that choice.

The kind of retrocausation discussed above is sometimes invoked to explain EPR non-locality, but mostly as a curiosity because it is believed to be of no consequence. It has no consequences also in the representation of the quantum algorithm with respect to Bob and any external observer. To them, it leaves the input state of the algorithm – state (7) – unaltered. It just tells that, say, the left digit of the random outcome of Bob's measurement $\mathbf{b} = 10$ has been randomly selected by Bob's measurement and the right digit has been randomly selected back in time by the future Alice's measurement – in fact an inconsequential thing.

Things change dramatically in the representation with respect to Alice – the problem solver.

We have seen that the projection induced by Alice's measurement of $\hat{B}_1$ in the output state (10) must propagate backward in time through the inverse of $H_A U_f H_A U_B$ until t_0, where it selects the right digit of the random outcome of Bob's measurement 10. Let us see the value of this backward propagation at time t_1, immediately after the application of U_B and before that of $H_A U_f H_A$. This time we should advance the two ends of the projection of state (10) on state (14) by the inverse of $H_A U_f H_A$. The result is the projection of state (9), the input state of the quantum algorithm with respect to Alice, on:

$$H_A^\dagger U_f^\dagger H_A^\dagger \left|\chi\right\rangle = \frac{1}{2}\left(\mathrm{e}^{i\varphi_1}\left|01\right\rangle_B + \mathrm{e}^{i\varphi_3}\left|11\right\rangle_B\right)\left|0\right\rangle_A\left(\left|0\right\rangle_V - \left|1\right\rangle_V\right). \qquad (17)$$

This is an outstanding consequence. State (17), the input state to Alice under the assumption that the selection of the solution equally shares between Bob's and Alice's measurements, tells her, before she performs any function evaluation, that the suffix of the function chosen by Bob is either $\mathbf{b} = 01$ or $\mathbf{b} = 11$, namely that $\mathbf{b} \in \{01, 11\}$. We can say that Alice *knows in advance* that $\mathbf{b} \in \{01, 11\}$, since this knowledge comes from the projection of the quantum state induced by her future measurement.

We provide the following interpretation of this advanced knowledge. We are at a fundamental level where knowing is doing [28]. Alice is the problem solver,

her knowing in advance that $\mathbf{b} \in \{01, 11\}$ would simply mean that the quantum algorithm requires the number of function evaluations logically required to identify the solution starting from that knowledge. We mean by classical logic. This of course establishes a correspondence between quantum computation and classical logic. It is the main assumption of the present work.

In the present case, the number of function evaluations required to discriminate between $f_{01}(\mathbf{a})$ and $f_{11}(\mathbf{a})$ is just one. In fact the value of the function for the argument $\mathbf{a} = 0$ does the job – see the tables of the two functions in question in array (1). Since it is 0, the function must be $f_{01}(\mathbf{a})$ This implies that it is balanced.

With problem setting $\mathbf{b} = 01$, there are three instances of Alice's advanced knowledge: $\mathbf{b} \in \{01, 00\}_B$, $\mathbf{b} \in \{01, 11\}_B$, and $\mathbf{b} \in \{01, 10\}_B$. As one can see, $\mathbf{b} = 01$ goes with each one of the other three possible values of $\mathbf{b}$. Similarly for problem setting $\mathbf{b} = 00$, etc. It turns out that the quantum algorithm can be seen as a sum over classical histories in each of which Alice knows in advance that the value of $\mathbf{b}$ chosen by Bob is either one of a particular pair of values and performs the function evaluation for the value of the argument that tells which one.

Let us see this in more detail. We see the quantum algorithm under the perspective of Feynman's sum over classical histories [29]. A classical history is a classical trajectory of the quantum registers, namely a causal sequence of sharp register states. For example:

$$e^{i\varphi_1} |01\rangle_B |0\rangle_A |0\rangle_V \overset{H_A}{\to} e^{i\varphi_1} |01\rangle_B |0\rangle_A |0\rangle_V \overset{U_f}{\to} e^{i\varphi_1} |01\rangle_B |0\rangle_A |0\rangle_V \overset{H_A}{\to} e^{i\varphi_1} |01\rangle_B |1\rangle_A |0\rangle_V . \tag{18}$$

The left-most state is one of the elements of the input state superposition (9). The state after each arrow is one of the elements of the superposition generated by the unitary transformation of the state before the arrow; the transformation in question is specified above the arrow.

In history (18), the problem setting is $\mathbf{b} = 01$. Alice performs function evaluation for $\mathbf{a} = 0$ (second and third state). This behavior is justifiable by two instances of Alice's advanced knowledge. One is $\mathbf{b} \in \{01, 11\}_B$, the other $\mathbf{b} \in \{01, 10\}_B$. The value of the function for $\mathbf{a} = 0$ in either case tells that the function in the black box is $f_{01}(\mathbf{a})$ and thus that it is balanced.

Summing up, with $R = \frac{1}{2}$, the quantum algorithm can be seen as a sum over classical histories in each of which Alice knows in advance one of the possible halves of the problem setting and performs the function evaluations logically required to identify the missing half and thus the solution.

5 Generalization

We show that, given an oracle problem, we can know the number of function evaluations required to solve it with quantum retrocausality $R = \frac{1}{2}$. A generic oracle problem can be formulated as follows. We have a set of functions $f_{\mathbf{b}} : \{0,1\}^n \to \{0,1\}^m$ with $m \leq n$. The suffix $\mathbf{b}$ ranges over the set of all the

problem settings σ_B. Bob chooses one of these functions (a value of $\mathbf{b}$) and gives Alice the black box (oracle) that computes it. Alice knows the set of functions but does not know which is the function chosen by Bob. She is to find a certain feature of the function (eg whether it is constant or balanced in the algorithm of Deutsch, or its period in that of Shor) by performing function evaluations (oracle queries). We call the feature in question, which is the solution of the problem and a function of $\mathbf{b}$, $s(\mathbf{b})$.

5.1 Time-symmetric representation to Alice

Provided that a register B contains the problem setting $\mathbf{b}$ and a register A will eventually contain the solution $s(\mathbf{b})$, the most general form of the input and output states of the unitary part (U) of Alice's problem-solving action, in the representation of the quantum algorithm to her, is:

$$|\text{in}\rangle_{BAW} = \frac{1}{\sqrt{c}} \left(\sum_{\mathbf{b} \in \sigma_B} \mathrm{e}^{i\varphi_{\mathbf{b}}} |\mathbf{b}\rangle_B \right) |00...\rangle_A |\psi\rangle_W , \tag{19}$$

$$|\text{out}\rangle_{BAW} = U |\text{in}\rangle_{BAW} = \frac{1}{\sqrt{c}} \sum_{\mathbf{b} \in \sigma_B} \mathrm{e}^{i\varphi_{\mathbf{b}}} |\mathbf{b}\rangle_B |s(\mathbf{b})\rangle_A |\varphi(\mathbf{b})\rangle_W , \tag{20}$$

where c is the cardinality of σ_B, $|\psi\rangle_W$ and $|\varphi(\mathbf{b})\rangle_W$ are normalized states of a register W, which stands for any other register or set of registers.

U should not change the problem setting. It suffices that register B is the control register of all function evaluations, what means that the content of register B affects the output of function evaluation while remaining unaltered through it, and the unitary transformations before and after each function evaluation do not apply to B. Correspondingly, U sends the input into the output independently term by term and keeping the value of $\mathbf{b}$ unaltered:

$$\forall \mathbf{b}: \quad U |\mathbf{b}\rangle_B |00...\rangle_A |\psi\rangle_W = |\mathbf{b}\rangle_B |s(\mathbf{b})\rangle_A |\varphi(\mathbf{b})\rangle_W . \tag{21}$$

Given the oracle problem, namely all the pairs $\mathbf{b}$ and $s(\mathbf{b})$, and provided that one is free to add suitable *garbage qubits* to register W, it should not be difficult to put the input and output states in a form compatible with the existence of such a U between them. In the following, we assume that states (19) and (20) are of this form. We will see that we do not need to know the form of U to the end of ascertaining the number of function evaluations required to solve the oracle problem with quantum retrocausality $R = \frac{1}{2}$; it suffices to know all the pairs $\mathbf{b}$ and $s(\mathbf{b})$.

Note that, for equation (21), the projection of the quantum state induced by any measurement on the content of register B in the output state, advanced by $U^\dagger$, becomes the projection induced by performing the same measurement in the input state. Conversely, the projection induced by any measurement on the content of B in the input state, retarded by U, becomes the projection induced by performing the same measurement in the output state. This goes

along with the fact that the reduced density operator of register B remains the same throughout U. Its random phase representation is

$$|\psi\rangle_B = \frac{1}{\sqrt{c}} \sum_{\mathbf{b} \in \sigma_B} \mathrm{e}^{i\varphi_{\mathbf{b}}} |\mathbf{b}\rangle_B \,, \tag{22}$$

the usual representation being $\rho_B = \frac{1}{c} \sum_{\mathbf{b} \in \sigma_B} |\mathbf{b}\rangle \langle \mathbf{b}|_B$.

5.2 Quantum feedback

Given the equations (19) and (20), we show how to share the selection of the solution between initial and final measurements and derive the corresponding Alice's advanced knowledge.

It is simpler to assume that U_B is the identity. In this way we can think that the initial Bob's measurement is performed in state (19). Of course its selection of a value of $\mathbf{b}$ also determines that of $s(\mathbf{b})$.

We reformulate Occam conditions **1** and **2** for the particular case $R = \frac{1}{2}$. We should reduce in all the possible ways the two measurements of $\hat{B}$, one on the part of Bob in the input state and the other on the part of Alice in the output state (see Section 4.2) to two partial measurements – of $\hat{B}_i$ and $\hat{B}_j$ – such that:

I together, they select whatever is selected by the complete measurements and

II each performed alone, they contribute in an equal and non-redundant way to the selection of the solution.

Let $\mathcal{E}_A$ be the von Neumann entropy of the solution, namely of the trace over registers B and W of state $|\mathrm{out}\rangle_{BAW}$. Point **II** implies the following two conditions:

$$\Delta\mathcal{E}_A\left(\hat{B}_i\right) = \Delta\mathcal{E}_A\left(\hat{B}_j\right), \tag{23}$$

where $\Delta\mathcal{E}_A\left(\hat{B}_i\right)$ is the reduction of $\mathcal{E}_A$ due to the measurement of $\hat{B}_i$, $\Delta\mathcal{E}_A\left(\hat{B}_j\right)$ that due to the measurement of $\hat{B}_j$, and:

$$\boxed{\textit{no partial measurement outcome provides enough information to select the solution.}} \tag{24}$$

In fact the cases are two: if both outcomes provided enough information, then there would be redundant information, what is forbidden by the no-redundancy condition. If only one did, then the two partial measurements would not contribute equally to the selection of the solution, what is forbidden by the equality condition. Condition (24) is redundant when $\mathbf{b}$ is an unstructured bit string as in Deutsch algorithm, it is not when $\mathbf{b}$ is structured.

Alice's measurement of $\hat{B}_j$ (as any measurement of a Boolean function of $\hat{B}$), performed alone, must induce a projection of the output state (20) on a state of the general form

$$|\chi\rangle = \frac{1}{\sqrt{c'}} \sum_{\mathbf{b} \in \sigma'_B} \mathrm{e}^{i\varphi_{\mathbf{b}}} |\mathbf{b}\rangle_B |s(\mathbf{b})\rangle_A |\varphi(\mathbf{b})\rangle_W ,$$

where σ'_B is a subset of σ_B of cardinality c'. Alice's advanced knowledge is obtained by advancing by $U^\dagger$ the two ends of this projection at the input of the quantum algorithm, at time t_1 immediately after the preparation of the problem setting (which here is the outcome of Bob's measurement). Even without knowing $U^\dagger$, we know that, for equation (21), this projects the input state (19) on:

$$\frac{1}{\sqrt{c'}} \left(\sum_{\mathbf{b} \in \sigma'_B} \mathrm{e}^{i\varphi_{\mathbf{b}}} |\mathbf{b}\rangle_B \right) |00...\rangle_A |\psi\rangle_W . \tag{25}$$

In particular, it projects the maximally mixed state of register B (22) on the state of lower entropy

$$\frac{1}{\sqrt{c'}} \sum_{\mathbf{b} \in \sigma'_B} \mathrm{e}^{i\varphi_{\mathbf{b}}} |\mathbf{b}\rangle_B , \tag{26}$$

which represents Alice's advanced knowledge – Alice knows in advance that $\mathbf{b} \in \sigma'_B$. For short we say that Alice's measurement of $\hat{B}_j$ projects σ_B on σ'_B.

Still for equation (21), the same projection can be obtained by measuring $\hat{B}_j$ in the input state. We also note that, mathematically, nothing changes if we assume to start with that the two complete measurements reduce to two partial measurements, of $\hat{B}_i$ and $\hat{B}_j$, both performed in the input state. Conditions **I** and **II** define the same pairs of partial observables $\hat{B}_i$ and $\hat{B}_j$ no matter whether $\hat{B}_j$ is measured in the input or output state. In fact moving the measurement from the output to the input state leaves all selections and reductions of the entropy of the solution unaltered.

This latter way of assessing Alice's advanced knowledge highlights a symmetry hidden in the former one. We are left with two partial measurements of the content of register B that satisfy conditions **I** and **II**, both performed in the input state. We can loose the memory of which partial measurement is performed by Alice and which by Bob. Evidently, either partial measurement can be the one performed by Alice. Therefore, given a pair of partial measurements, of $\hat{B}_i$ and $\hat{B}_j$, in the input state (19) that satisfy conditions **I** and **II**, either partial measurement performed alone projects the maximally mixed state of register B (22) on an instance of Alice's advanced knowledge. By the way, in this sense we can say that, with quantum retrocausality $\mathcal{R} = \frac{1}{2}$, Alice knows "half" of the problem setting in advance.

It is important to note that register W, which we have considered for generality and could be necessary to construct the quantum algorithm, is not involved in the definition of the pairs $\hat{B}_i$ and $\hat{B}_j$. Let us recall the conditions their measurements [which can be both performed in the input state (19)] are submitted

to: (i) together, they select a value of $\mathbf{b}$, (ii) the information acquired by either measurement is not acquired by the other, (iii) they satisfy equation (23), and (iv) they satisfy requirement (24). Conditions (i), (ii) and (iv) only involve the input state of register B, namely state (22). Also condition (iii) does not involve register W, as the reductions of the entropy of the solution $\Delta\mathcal{E}_A\left(\hat{B}_i\right)$ and $\Delta\mathcal{E}_A\left(\hat{B}_j\right)$ concern the trace of the output state (20) over registers B and W.

Therefore, to the end of determining $\hat{B}_i$ and $\hat{B}_j$, we can work with $|\text{in}\rangle_{BA}$ and $|\text{out}\rangle_{BA}$, the traces over W of $|\text{in}\rangle_{BAW}$ and $|\text{out}\rangle_{BAW}$; it suffices to drop $|\psi\rangle_W$ and the $|\varphi(\mathbf{b})\rangle_W$. States $|\text{in}\rangle_{BA}$ and $|\text{out}\rangle_{BA}$, in turn, can be written solely on the basis of the pairs $\mathbf{b}$ and $s(\mathbf{b})$, namely of the oracle problem.

Since the quantum algorithm can be seen as a sum over classical histories in each of which Alice knows in advance one of the possible halves of the problem setting and performs the function evaluations still necessary to identify the solution, given an oracle problem, we can know the number of function evaluations required to solve it with quantum retrocausality $R = \frac{1}{2}$.

5.2.1 Example of application

We apply the present procedure to Deutsch's problem. Of course we should ignore Deutsch algorithm.

Given the problem, namely all the pairs $\mathbf{b}$ (ranging over $00, 01, 10, 11$) and $s(\mathbf{b})$ (0 if the function is constant, 1 if it is balanced), we write down $|\text{in}\rangle_{BA}$ and $|\text{out}\rangle_{BA}$ [of course we obtain the traces over register V of states (9) and (10)]. We do not need to know U. It suffices to know that there can be a unitary transformation between input and output that satisfies equation (21). Under conditions **I** and **II**, $|\text{in}\rangle_{BA}$ and $|\text{out}\rangle_{BA}$ define the pairs of partial observables $\hat{B}_i$ and $\hat{B}_j$ we are looking for – it is easier to think they are both measured in $|\text{in}\rangle_{BA}$. It is not a constructive definition, however finding the pairs in question will be easy in all the cases examined in this work. In the case of Deutsch's problem they are any two of the three partial observables: $\hat{B}_0$, $\hat{B}_1$, and $\hat{B}_X \equiv \text{XOR}\left(\hat{B}_0, \hat{B}_1\right)$. These partial observables are Boolean functions of $\hat{B}$ and the measurements of any two of them satisfy conditions **I** and **II** with $\Delta\mathcal{E}_A\left(\hat{B}_i\right) = \Delta\mathcal{E}_A\left(\hat{B}_j\right) = 1/2$ bit.

Given a problem setting, say $\mathbf{b} = 01$, either partial observable, $\hat{B}_i$ or $\hat{B}_j$, corresponds to an instance of Alice's advanced knowledge as follows. We should assume that its measurement selects the eigenvalue that matches with the problem setting. With problem setting $\mathbf{b} \equiv b_0 b_1 = 01$, this implies that the measurement of $\hat{B}_0$ selects $b_0 = 0$, that of $\hat{B}_1$ selects $b_1 = 1$, and that of $\hat{B}_X$ selects $\text{XOR}(b_0, b_1) = 1$. The corresponding projections of σ_B are respectively on $\{00, 01\}_B$, $\{01, 11\}_B$, and $\{01, 10\}_B$. Thus the instances of Alice's advanced knowledge are $\mathbf{b} \in \{01, 00\}_B$, $\mathbf{b} \in \{01, 11\}_B$, and $\mathbf{b} \in \{01, 10\}_B$, as obvious in hindsight. For any of these instances, Alice can solve the problem with a single function evaluation.

We call the present procedure *the advanced knowledge rule*. Given a generic oracle problem, this rule defines the number of function evaluations required to solve it with quantum retrocausality $\mathcal{R} = \frac{1}{2}$. The importance of this rule depends on the confidence that can be placed in the assumption that retrocausality $R = \frac{1}{2}$ is always attainable. This is the case in all the quantum algorithms examined in the present work. Whether it is the case in general should be the object of further work, the present one is an exploration.

6 Grover Algorithm

Bob hides a ball in one of N drawers (ie, he marks an item in an unstructured database of size N). Alice is to locate it by opening drawers. In the classical case, to be a-priori certain of locating the ball, Alice should plan to open $\mathrm{O}(N)$ drawers, in the case of Grover [30] quantum search algorithm $\mathrm{O}\left(\sqrt{N}\right)$.

The problem, an oracle one, is formalized as follows. Let $\mathbf{b}$ and $\mathbf{a}$, belonging to $\{0,1\}^n$, with $2^n = N$, be respectively the number of the drawer with the ball and that of the drawer that Alice wants to open. Checking whether the ball is in drawer $\mathbf{a}$ amounts to evaluating the function $f_{\mathbf{b}}(\mathbf{a}) : \{0,1\}^n \to \{0,1\}$, which is 1 if $\mathbf{a} = \mathbf{b}$ and 0 otherwise.

Bob chooses one of the functions $f_{\mathbf{b}}(\mathbf{a})$ (ie a value of $\mathbf{b}$) and gives Alice the black box that computes it. Alice is to find the value of $\mathbf{b}$ chosen by Bob by performing function evaluations for appropriate values of $\mathbf{a}$.

We will distinguish between $n = 2$ and $n > 2$. The speedup of Grover's algorithm with $n = 2$ is explained by $R = \frac{1}{2}$. When n goes past 2, R slightly goes above $\frac{1}{2}$, to go back to $\frac{1}{2}$ for $n \to \infty$.

6.1 Grover algorithm with $n = 2$

6.1.1 Time-symmetric representation to Alice

The input and output states of the quantum algorithm to Alice are respectively:

$$U_B |\psi\rangle = |\psi\rangle = \frac{1}{2\sqrt{2}} \left(\mathrm{e}^{i\varphi_0} |00\rangle_B + \mathrm{e}^{i\varphi_1} |01\rangle_B + \mathrm{e}^{i\varphi_2} |10\rangle_B + \mathrm{e}^{i\varphi_3} |11\rangle_B\right) |00\rangle_A \left(|0\rangle_V - |1\rangle_V\right), \tag{27}$$

$$\begin{aligned} \Im_A U_f H_A U_B |\psi\rangle &= \frac{1}{2\sqrt{2}} \left(\mathrm{e}^{i\varphi_0} |00\rangle_B |00\rangle_A + \mathrm{e}^{i\varphi_1} |01\rangle_B |01\rangle_A + \mathrm{e}^{i\varphi_2} |10\rangle_B |10\rangle_A + \mathrm{e}^{i\varphi_3} |11\rangle_B |11\rangle_A\right) \\ &\quad \left(|0\rangle_V - |1\rangle_V\right). \end{aligned} \tag{28}$$

The function of registers B, A, and V is as in Deutsch algorithm. U_B unitarily transforms the random outcome of Bob's measurement into the desired problem setting, H_A is the Hadamard transform on register A, U_f is function evaluation, and $\Im_A$ – a unitary transformation on register A – is the so called *inversion about the mean*. Note that we could write the input and output states of registers B and A only on the basis of the pairs $\mathbf{b}$ and $s(\mathbf{b})$ and without knowing Grover

algorithm. The state of register V is not relevant for the determination of Alice's advanced knowledge.

Measuring $\hat{A}$ in the output state (28) yields the number of the drawer with the ball chosen by Bob.

6.1.2 Quantum feedback

We apply the advanced knowledge rule to Grover's problem with $n = 2$. This yields the number of function evaluations required to solve the problem with quantum retrocausality $R = \frac{1}{2}$. The pairs of partial observables are the same as in Deutsch algorithm: all the pairs among $\hat{B}_0$, $\hat{B}_1$, and $\hat{B}_X$. One can see that they satisfy conditions **I** and **II** with $\Delta\mathcal{E}_A\left(\hat{B}_i\right) = \Delta\mathcal{E}_A\left(\hat{B}_j\right) = 1$ bit.

Say that the problem setting chosen by Bob is $\mathbf{b} = 01$ – ie Bob hides the ball in drawer 01. The instances of Alice's advanced knowledge are: $\mathbf{b} \in \{01, 00\}_B$, $\mathbf{b} \in \{01, 11\}_B$, and $\mathbf{b} \in \{01, 10\}_B$. In other words, Alice knows in advance that the ball is in one of a pair drawers (one of which with the ball in it). This allows her to locate the ball by opening either drawer (ie by performing just one function evaluation).

All the above could be derived solely from $|\text{in}\rangle_{BA}$ and $|\text{out}\rangle_{BA}$, the traces over register V of states (27) and (28), which can be written solely on the basis of the pairs $\mathbf{b}$ and $s(\mathbf{b})$. One does not need to know Grover algorithm. However, it is of course in agreement with the $n = 2$ instance of Grover algorithm. This means that the speedup of this instance is explained by quantum retrocausality $R = \frac{1}{2}$.

We check that the present instance of Grover algorithm can be seen as a sum over classical histories in each of which Alice knows in advance that the ball is in a pair of drawers and locates it by opening either drawer. A history is for example:

$$e^{i\varphi_1}|01\rangle_B|00\rangle_A|0\rangle_V \xrightarrow{H_A} e^{i\varphi_1}|01\rangle_B|11\rangle_A|0\rangle_V \xrightarrow{U_f} e^{i\varphi_1}|01\rangle_B|11\rangle_A|0\rangle_V \xrightarrow{\Im_A} e^{i\varphi_1}|01\rangle_B|01\rangle_A|0\rangle_V . \tag{29}$$

The problem setting is $\mathbf{b} = 01$. Alice performs function evaluation for $\mathbf{a} = 11$ (second and third state). Therefore we must assume that Alice's advanced knowledge is $\mathbf{b} \in \{01, 11\}_B$. Since the output of function evaluation is zero (the content of register V remains unaltered), she finds that the number of the drawer with the ball must be $\mathbf{b} = 01$.

6.2 Grover algorithm with $n > 2$

We should make a clarification to start with. With $n > 2$, the original Grover algorithm does not provide the solution of the problem with absolute certainty. For this, one has to resort to the revisitation of Grover algorithm made by Long [31] – see also [32]. Long's algorithm can be tuned to provide the solution of Grover's problem with certainty with any number of function evaluations provided it is above the minimum number required by the optimal quantum algorithm, which is $K = \frac{\pi}{4 \arcsin 2^{-n/2}} \approx \frac{\pi}{4} 2^{n/2}$. Incidentally, this is also the number

required by Grover algorithm, which however does not provide the solution with certainty when $n > 2$.

With $\mathcal{R} = \frac{1}{2}$, the number of function evaluations required by the present retrocausality model would be $2^{n/2} - 1 \approx 2^{n/2}$. In fact, Alice knows in advance $\mathcal{R}n$ of the n bits specify the number of the drawer with the ball, thus (with $\mathcal{R} = \frac{1}{2}$) $n/2$ bits. This means that she must open in the worst case $2^{n/2} - 1$ drawers (if all were empty, then she would know that the ball is in the only drawer left).

We note anyhow that also the number of function evaluations foreseen by the advanced knowledge rule, for $\mathcal{R} = \frac{1}{2}$, is that of an existing quantum algorithm, which is in fact Long's algorithm tuned on $2^{n/2} - 1$ function evaluations.

When n goes past 2, Alice's advanced knowledge should increase over the $n/2$ bits of the case $\mathcal{R} = \frac{1}{2}$, so that the problem can be solved with $\approx \frac{\pi}{4} 2^{n/2}$ function evaluations rather than $\approx 2^{n/2}$. This increase must be slight: an increase of just one bit would halve the required number of function evaluations. Correspondingly, $\mathcal{R}$ should slightly go above $\frac{1}{2}$. It should also be noted that, for $n \to \infty$, we have $\mathcal{R} = \frac{1}{2}$ again.

7 Deutsch&Jozsa algorithm

Deutsch&Jozsa [33] algorithm is a generalization of the seminal Deutsch algorithm that yields an exponential speedup. In the respective problem, the set of functions is all the constant and *balanced* functions (with the same number of zeroes and ones) $f_{\mathbf{b}} : \{0,1\}^n \to \{0,1\}$. Array (30) gives the tables of four of the eight functions for $n = 2$.

$\mathbf{a}$	$f_{0000}(\mathbf{a})$	$f_{1111}(\mathbf{a})$	$f_{0011}(\mathbf{a})$	$f_{1100}(\mathbf{a})$	...
00	0	1	0	1	...
01	0	1	0	1	...
10	0	1	1	0	...
11	0	1	1	0	...

(30)

The bit string $\mathbf{b} \equiv b_0 b_1 ... b_{2^n - 1}$ is both the suffix and the table of the function $f_{\mathbf{b}}(\mathbf{a})$ – the sequence of function values for increasing values of the argument. Alice is to find whether the function chosen by Bob is constant or balanced by computing $f_{\mathbf{b}}(\mathbf{a})$ for appropriate values of $\mathbf{a}$. Classically, this requires in the worst case a number of function evaluations exponential in n. It requires just one function evaluation in the quantum case.

7.1 Time-symmetric representation to Alice

The input and output states of the quantum algorithm to Alice are respectively:

$$U_B |\psi\rangle = |\psi\rangle = \frac{1}{4} \left(e^{i\varphi_0} |0000\rangle_B + e^{i\varphi_1} |1111\rangle_B + e^{i\varphi_2} |0011\rangle_B + e^{i\varphi_3} |1100\rangle_B + ... \right) |00\rangle_A \left(|0\rangle_V - |1\rangle_V \right),$$

$$H_A U_f H_A U_B |\psi\rangle = \frac{1}{4}\left[\left(e^{i\varphi_0}|0000\rangle_B - e^{i\varphi_1}|1111\rangle_B\right)|00\rangle_A + \left(e^{i\varphi_2}|0011\rangle_B - e^{i\varphi_3}|1100\rangle_B\right)|10\rangle_A + ...\right] (|0\rangle_V - |1\rangle_V). \quad (31)$$

Registers B, A, and V and the unitary transformation U_B have the same function as in the previous quantum algorithms. H_A is the Hadamard transform on register A and U_f is function evaluation. Note that we could have written the input and output states of registers B and A only on the basis of the pairs $\mathbf{b}$ and $s(\mathbf{b})$.

Measuring $\hat{A}$ in the output state (31) says that the function is constant if the measurement outcome is all zeros, balanced otherwise.

7.2 Quantum feedback

We apply the advanced knowledge rule to Deutsch&Jozsa's problem. Given the problem setting of a balanced function, there is only one pair of partial measurements of the content of register B compatible with conditions **I** and **II**. With problem setting, say, $\mathbf{b} = 0011$, $\hat{B}_i$ must be the content of the left half of register B and $\hat{B}_j$ that of the right half. The measurement of $\hat{B}_i$ yields all zeros, that of $\hat{B}_j$ all ones.

In fact, a partial measurement yielding both zeroes and ones would violate condition (24): it would provide enough information to identify the solution – the fact that $f_{\mathbf{b}}$ is balanced. Given that either partial measurement must yield all zeroes or all ones, it must concern the content of half register. Otherwise either equation (23) would be violated or the problem setting would not be completely determined, as readily checked.

One can see that, with $\mathbf{b} = 0011$, the measurement of $\hat{B}_i$, performed alone, projects σ_B on the subset $\{0011, 0000\}_B$, that of $\hat{B}_j$ on $\{0011, 1111\}_B$. Either subset represents the part of the problem setting that Alice knows in advance. Equation (23) is satisfied with $\Delta\mathcal{E}_A\left(\hat{B}_i\right) = \Delta\mathcal{E}_A\left(\hat{B}_j\right) = 1$ bit.

The case of the problem setting of a constant function is analogous. The only difference is that there are more pairs of partial measurements that satisfy the above said conditions. Say that the problem setting is $\mathbf{b} = 0000$. The measurements of the content of the left and right half of register B (each performed alone) projects σ_B on respectively $\{0000, 0011\}_B$ and $\{0000, 1100\}_B$, the measurements of the content of even and odd cells (say the leftmost one is odd) on respectively $\{0000, 0101\}_B$ and $\{0000, 1010\}_B$, etc.

There is a shortcut to finding the subsets in question. Here the problem setting – the bit string $\mathbf{b}$ – is the table of the function chosen by Bob. For example $\mathbf{b} = 0011$ is the table $f_{\mathbf{b}}(00) = 0, f_{\mathbf{b}}(01) = 0, f_{\mathbf{b}}(10) = 1, f_{\mathbf{b}}(11) = 1$. We call "good half table" any half table in which all the values of the function are the same. One can see that good half tables are in one-to-one correspondence with the subsets of σ_B in question. For example, the good half table $f_{\mathbf{b}}(00) = 0, f_{\mathbf{b}}(01) = 0$ corresponds to the subset $\{0011, 0000\}_B$, is the identical part of the two bit-strings in it. Thus, given a problem setting, ie an entire table, either

good half table, or identically the corresponding subset of σ_B, is a possible instance of Alice's advanced knowledge.

Because of the structure of tables, given the advanced knowledge of a good half table, the entire table and thus the solution can be identified by performing just one function evaluation for any value of the argument $\mathbf{a}$ outside the half table.

Summing up, the advanced knowledge rule says that Deutsch&Jozsa's problem can be solved with just one function evaluation. This is in agreement with Deutsch&Jozsa algorithm, what also means that the speedup of this algorithm is explained by quantum retrocausality $\mathcal{R} = \frac{1}{2}$.

We check that the present instance of Deutsch&Jozsa algorithm can be seen as a sum over classical histories in each of which Alice knows in advance that Bob has chosen one of a pair of functions and discriminates between the two with just one function evaluation. A history is for example: $e^{i\varphi_2} |0011\rangle_B |00\rangle_A |0\rangle_V \overset{H_A}{\to} e^{i\varphi_2} |0011\rangle_B |10\rangle_A |0\rangle_V \overset{U_f}{\to} e^{i\varphi_2} |0011\rangle_B |10\rangle_A |1\rangle_V \overset{H_A}{\to} e^{i\varphi_2} |0011\rangle_B |10\rangle_A |1\rangle_V$. Since the problem setting is $\mathbf{b} = 0011$ and Alice performs function evaluation for $\mathbf{a} = 10$, her advanced knowledge must be $\mathbf{b} \in \{0011, 0000\}_B$; if it were $\mathbf{b} \in \{0011, 1111\}_B$, she would have performed function evaluation for either $\mathbf{a} = 00$ or $\mathbf{a} = 01$. The result of function evaluation, $f_{\mathbf{b}}(10) = 1$, tells that the function chosen by Bob is $f_{0011}(\mathbf{a})$, hence that it is balanced.

One can see that the present analysis, like the notion of good half table, holds unaltered for $n > 2$.

8 Simon and hidden subgroup algorithms

In Simon's [34] problem, the set of functions is all the $f_{\mathbf{b}} : \{0,1\}^n \to \{0,1\}^{n-1}$ such that $f_{\mathbf{b}}(\mathbf{a}) = f_{\mathbf{b}}(\mathbf{c})$ if and only if $\mathbf{a} = \mathbf{c}$ or $\mathbf{a} = \mathbf{c} \oplus \mathbf{h}(\mathbf{b})$; $\oplus$ denotes bitwise modulo 2 addition. The bit string $\mathbf{h}(\mathbf{b})$, depending on $\mathbf{b}$, is a sort of period of the function.

Array (32) gives the tables of four of the six functions for $n = 2$. The bit string $\mathbf{b}$ is both the suffix and the table of the function. We note that each value of the function appears exactly twice in each table; thus 50% of the rows plus one always identify $\mathbf{h}(\mathbf{b})$.

	$\mathbf{h}(0011) = 01$	$\mathbf{h}(1100) = 01$	$\mathbf{h}(0101) = 10$	$\mathbf{h}(1010) = 10$	...
$\mathbf{a}$	$f_{0011}(\mathbf{a})$	$f_{1100}(\mathbf{a})$	$f_{0101}(\mathbf{a})$	$f_{1010}(\mathbf{a})$	...
00	0	1	0	1	...
01	0	1	1	0	...
10	1	0	0	1	...
11	1	0	1	0	...

(32)

Bob chooses one of these functions. Alice is to find the value of $\mathbf{h}(\mathbf{b})$ by performing function evaluation for appropriate values of $\mathbf{a}$.

In present knowledge, a classical algorithm requires a number of function evaluations exponential in n. The quantum part of Simon algorithm solves with just one function evaluation the hard part of this problem, namely finding a string $\mathbf{s}_j(\mathbf{b})$ *orthogonal* [34] to $\mathbf{h}(\mathbf{b})$. There are 2^{n-1} such strings. Running the quantum part yields one of these strings at random. The quantum part is iterated until finding $n-1$ different strings. This allows Alice to find $\mathbf{h}(\mathbf{b})$ by solving a system of modulo 2 linear equations. Thus, on average, finding $\mathbf{h}(\mathbf{b})$ requires $\mathrm{O}(n)$ iterations of the quantum part – in particular $\mathrm{O}(n)$ function evaluations. Moreover, if we put an upper bound to the number of iterations, a-priori there is always a non-zero probability of not finding $n-1$ different strings.

We apply the advanced knowledge rule directly to the complete Simon's problem of finding $\mathbf{h}(\mathbf{b})$ through function evaluations. This is not the problem solved by the quantum part of Simon algorithm, which is finding at random one of the $\mathbf{s}_j(\mathbf{b})$ orthogonal to $\mathbf{h}(\mathbf{b})$. The value of $\mathcal{R}$ that explains the speedup of the quantum part of Simon algorithm will be a by-product of applying the advanced knowledge rule to Simon's problem.

8.1 Time-symmetric representation to Alice

Knowing all the pairs $\mathbf{b}$, $\mathbf{h}(\mathbf{b})$ – from array (32) – we can write $|\mathrm{in}\rangle_{BA}$ and $|\mathrm{out}\rangle_{BA}$:

$$|\mathrm{in}\rangle_{BA} = \frac{1}{\sqrt{6}}\left(\mathrm{e}^{i\varphi_0}|0011\rangle_B + \mathrm{e}^{i\varphi_1}|1100\rangle_B + \mathrm{e}^{i\varphi_2}|0101\rangle_B + \mathrm{e}^{i\varphi_3}|1010\rangle_B + ...\right)|00\rangle_A,$$

$$|\mathrm{out}\rangle_{BA} = \frac{1}{\sqrt{6}}\left[\left(\mathrm{e}^{i\varphi_0}|0011\rangle_B + \mathrm{e}^{i\varphi_1}|1100\rangle_B\right)|01\rangle_A + \left(\mathrm{e}^{i\varphi_2}|0101\rangle_B + \mathrm{e}^{i\varphi_3}|1010\rangle_B\right)|10\rangle_A + ...\right].$$

We must assume that there can be a unitary transformation between the untraced states $|\mathrm{in}\rangle_{BAW}$ and $|\mathrm{out}\rangle_{BAW}$.

8.2 Quantum feedback

The analysis is similar to that of Deutsch&Jozsa algorithm. This time a good half table should not contain a same value of the function twice, what would provide enough information to identify the solution of the problem [ie the *period* $\mathbf{h}(\mathbf{b})$], thus violating condition (24) of the advanced knowledge rule.

With $\mathbf{b} = 0011$, namely $f_{\mathbf{b}}(00) = 0,\ f_{\mathbf{b}}(10) = 1, f_{\mathbf{b}}(01) = 0,\ f_{\mathbf{b}}(11) = 1$, one way of sharing the table into two good halves is: $f_{\mathbf{b}}(00) = 0,\ f_{\mathbf{b}}(10) = 1$ and $f_{\mathbf{b}}(01) = 0,\ f_{\mathbf{b}}(11) = 1$. The corresponding subsets of σ_B are respectively $\{0011, 0110\}_B$ and $\{0011, 1001\}_B$; one can check that each half table is the identical part of the two bit-strings in the corresponding subset of σ_B. Either good half table or identically either subset is a possible instance of Alice's advanced knowledge. Equation (23) is satisfied with $\Delta\mathcal{E}_A\left(\hat{B}_i\right) = \Delta\mathcal{E}_A\left(\hat{B}_j\right) = 0.585$ bit (entropy reduction from $-\log_2\frac{1}{3}$ bit to 1 bit).

We note parenthetically that sharing each table into two halves is accidental to Deutsch&Jozsa's and Simon algorithms. In the quantum part of Shor's [35] factorization algorithm (finding the period of a periodic function), taking two shares of the table that do not contain a same value of the function twice implies that each share is less than half table if the domain of the function spans more than two periods.

Given the advanced knowledge of a good half table, the entire table and then $\mathbf{h}(\mathbf{b})$ can always be identified by performing just one function evaluation for any value of the argument $\mathbf{a}$ outside the half table. Thus, the advanced knowledge rule says that, with $\mathcal{R} = \frac{1}{2}$, Simon's problem is solved with just one function evaluation. Under the assumption that $\mathcal{R} = \frac{1}{2}$ is always attainable, Simon algorithm, which requires $\mathrm{O}(n)$ function evaluations, would be suboptimal.

The above also shows that the speedup of the quantum part of Simon algorithm is explained by $\mathcal{R} = \frac{1}{2}$. In fact, once known $\mathbf{h}(\mathbf{b})$ – with just one function evaluation in the case of quantum retrocausality $\mathcal{R} = \frac{1}{2}$ – generating at random the $\mathbf{s}_j(\mathbf{b})$'s requires no further function evaluations.

We give the simplest instance, $n = 2$, of the quantum algorithm that finds $\mathbf{h}(\mathbf{b})$ with just one function evaluation. Register W reduces to the usual register V that contains the result of function evaluation modulo 2 added to its previous content. The input and output states of V are both $\frac{1}{\sqrt{2}}(|0\rangle_V - |1\rangle_V)$. We have $U = \mathcal{P}_A H_A U_f H_A$, where H_A is Hadamard on register A, U_f function evaluation, $\mathcal{P}_A$ the permutation of the basis vectors $|01\rangle_A$ and $|10\rangle_A$. Checking whether there is the similar algorithm for $n > 2$ should be the object of further work.

The sum over histories representation can be developed as in Deutsch&Jozsa algorithm. If, for example, Alice's advanced knowledge is $\mathbf{b} \in \{0011, 0110\}_B$, she can identify the value of $\mathbf{h}(\mathbf{b})$ by performing a single function evaluation for either $\mathbf{a} = 01$ or $\mathbf{a} = 11$ – see array (32) – etc.

The fact that Alice knows in advance a good half table, and can thus identify the entire table and hence the solution with just one function evaluation, clearly holds unaltered for $n > 2$. It should also apply to the generalized Simon's problem and to the Abelian hidden subgroup problem. In fact the corresponding algorithms are essentially Simon algorithm. In the hidden subgroup problem, the set of functions $f_{\mathbf{b}} : G \to W$ map a group G to some finite set W with the property that there exists some subgroup $S \leq G$ such that for any $\mathbf{a}, \mathbf{c} \in G$, $f_{\mathbf{b}}(\mathbf{a}) = f_{\mathbf{b}}(\mathbf{c})$ if and only if $\mathbf{a} + S = \mathbf{c} + S$. The problem is to find the hidden subgroup S by computing $f_{\mathbf{b}}(\mathbf{a})$ for the appropriate values of $\mathbf{a}$. Now, a large variety of problems solvable with a quantum speedup can be reformulated in terms of the hidden subgroup problem [11]. Among these we find: the seminal Deutsch's problem, finding orders, finding the period of a function (thus the problem solved by the quantum part of Shor's factorization algorithm), discrete logarithms in any group, hidden linear functions, self shift equivalent polynomials, Abelian stabilizer problem, graph automorphism problem [36].

9 Conclusion

We have extended the representation of the quantum algorithm to the process of setting the problem. The initial measurement, in a state where the problem setting is completely undetermined, selects a setting at random, a unitary evolution transforms it into the desired setting, a further unitary evolution solves the problem, and the final measurement reads the solution.

This extended representation would tell Alice, the problem solver, the setting of the problem before she begins her search for the solution. To Alice, this setting must be hidden inside the black box. To physically represent this concealment, we resorted to relational quantum mechanics. In the representation relativized to Alice, the projection induced by the initial measurement is retarded at the end of the unitary part of her problem solving action. To Alice, the setting remains completely undetermined throughout that part of her action.

In this time-symmetric representation of the quantum algorithm, the solution of the problem is selected by either the initial Bob's measurement or the final Alice's measurement. We assumed that the selection shares without redundancies between the two measurements. This turned out to be equivalent to sharing between initial and final measurement the selection of the random outcome of the initial measurement. We have called $\mathcal{R}$ the fraction of the information that specifies this random outcome whose selection is ascribed to the final measurement. $\mathcal{R}$ is a measure of retrocausality; $\mathcal{R} = 0$ means that the random outcome of the initial measurement is entirely selected by that same measurement, without retrocausality. $\mathcal{R} = 1$ means that it is entirely selected by the final measurement, without time-forward causality. $\mathcal{R} = \frac{1}{2}$ means that the selection equally shares between the two measurements.

The sharing in question is without consequences in the representation of the quantum algorithm with respect to Bob and any external observer, where it leaves the input state of the quantum algorithm unaltered. It projects the input state relativized to Alice, one of maximal ignorance of the problem setting, on a state of lower entropy where she knows the $\mathcal{R}$-the part of the problem setting in advance, before performing any function evaluation.

The quantum algorithm turns out to be a sum over classical histories in each of which Alice knows in advance one of the possible $\mathcal{R}$-th parts of the problem setting and performs the function evaluations logically required to identify the solution. The number of function evaluations is therefore that of a classical algorithm that benefits of the same advanced knowledge.

Given an oracle problem and a value of $\mathcal{R}$, the present explanation of the speedup provides the number of function evaluations required to solve it quantumly. Conversely, given a known quantum algorithm, it provides the value of $\mathcal{R}$ that explains its speedup.

We have compared this explanation of the speedup with the major quantum algorithms. $\mathcal{R} = \frac{1}{2}$ explains the speedup of the seminal Deutsch algorithm, of Grover quantum search algorithm for database size 4, Deutsch&Jozsa algorithm, and the algorithms of Simon and the Abelian hidden subgroup. All these algorithms require a single function evaluation.

When database size goes past 4, Grover algorithm requires more than one function evaluation and $\mathcal{R}$ goes slightly above $\frac{1}{2}$, going back to $\frac{1}{2}$ as database size goes to infinity.

In any way, the number of function evaluations foreseen by the present explanation with $\mathcal{R} = \frac{1}{2}$ is always that of an existing quantum algorithm and a good approximation of the number required by the optimal one.

If the sample of quantum algorithms examined were representative enough, namely if quantum retrocausality $\mathcal{R} = \frac{1}{2}$ were always attainable in quantum problem solving, we would have a very powerful tool for the study of quantum query complexity, a still open problem. This work is an exploration. Wether $\mathcal{R} = \frac{1}{2}$ is always attainable, what is the maximum value of $\mathcal{R}$ physically attainable are questions that remain open.

$\mathcal{R}$, the fraction of the information that specifies the random outcome of the initial measurement whose selection can mathematically be ascribed to the final measurement, would seem to be a potentially interesting retrocausality measure. Studying it from a foundational standpoint might be rewarding. Because of the fundamental character of quantum search in an unstructured database, one could conjecture that $\mathcal{R} = \frac{1}{2}$ is always attainable and that the maximum possible value of $\mathcal{R}$ is attained in Grover algorithm. Whether this is so and why should be the object of further work.

Another issue that might deserve further investigation is the possible relation between $\mathcal{R}$ and the information theoretic temporal Bell inequalities.

From a practical standpoint, one should further study the trust that can be placed in the validity of the $\mathcal{R} = \frac{1}{2}$ approximation by checking the relation between speedup and $\mathcal{R}$ on larger classes of known quantum algorithms. One could also investigate whether there is the optimal quantum algorithm foreseen in Section 8 for Simon's and the Abelian hidden subgroup problems.

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